

On Quantum Communication Channels with Constrained Inputs

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Abstract – The purpose of this work is to extend results of previous papers [13], [2] to quantum channels with additive constraints onto the input signal, by showing that the capacity of such channel is equal to the supremum of the entropy bound with respect to all apriori distributions satisfying the constraint. We also make an extension to channels with continuous alphabet. As an application, we prove the formula for the capacity of the quantum Gaussian channel with constrained energy of the signal, establishing the asymptotic equivalence of this channel to the quasiclassical photon channel, and derive the lower bounds for the reliability function of the pure-state Gaussian channel.

I. THE CASE OF DISCRETE ALPHABET

Most of the results concerning the capacity of quantum communication channels were proved for the case of finite input alphabets [9], [11], [7], [13]. The importance of considering channels with infinite (continuous) alphabets and with constrained inputs was clear from the beginning of quantum communications [6], [14], and was reiterated in [16]. The present paper, which is a continuation of our papers [13], [2], is devoted to study of this case.

For reader's convenience we start with repeating some basic notions from [13], making necessary modifications for channels with constrained inputs. Let \mathcal{H} be a Hilbert space. A quantum communication channel with (possibly infinite) discrete input alphabet $A = \{i\}$ consists of the mapping $i \rightarrow S_i$ from the input alphabet to the set of density operators (d. o.) in \mathcal{H} . The input is described by an *apriori* probability distribution $\pi = \{\pi_i\}$ on A . At the output there is a quantum measurement in the sense of [9], given by *resolution of identity* in \mathcal{H} , that is by a family $X = \{X_j\}$ of positive

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operators in \mathcal{H} satisfying $\sum_j X_j = I$, where I is the unit operator in \mathcal{H} . The index j runs through some discrete output alphabet, which is not fixed here. The conditional probability of the output j if the input was i equals to $P(j|i) = \text{Tr} S_i X_j$. The Shannon information is given by the classical formula

$$I_1(\pi, X) = \sum_j \sum_i \pi_i P(j|i) \log \left(\frac{P(j|i)}{\sum_k \pi_k P(j|k)} \right). \quad (1)$$

Let $f(i)$ be a function defined on the input alphabet. We shall restrict the apriori distributions π by the inequality

$$\sum_i f(i) \pi(i) \leq E, \quad (2)$$

where E is a real number, and denote the class of such probability distributions by \mathcal{P}_1 .

Let us consider also the product channel in the Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \dots \otimes \mathcal{H}$ with the input alphabet A^n consisting of words $u = (i_1, \dots, i_n)$ of length n , and the d. o.

$$S_u = S_{i_1} \otimes \dots \otimes S_{i_n}$$

corresponding to the word u . If π is a probability distribution on A^n and X is a resolution of identity in $\mathcal{H}^{\otimes n}$, we define the information quantity $I_n(\pi, X)$ by the formula similar to (1). We put the additive constraint onto the distribution π by asking

$$\sum_{i_1, \dots, i_n} [f(i_1) + \dots + f(i_n)] \pi(i_1, \dots, i_n) \leq nE, \quad (3)$$

and denote by \mathcal{P}_n the class of probability distributions satisfying this constraint.

Defining

$$C_n = \sup_{\pi \in \mathcal{P}_n; X} I_n(\pi, X), \quad (4)$$

we have the property of superadditivity $C_n + C_m \leq C_{n+m}$, hence the following limit exists

$$C = \lim_{n \rightarrow \infty} C_n/n, \quad (5)$$

which we call the *capacity* of the initial channel. (This quantity may be eventually infinite for infinite dimensional Hilbert space and in what follows

we consider the nontrivial case $C < \infty$). The definition is justified by the classical Shannon's coding theorem for channels with constrained inputs. Namely, call by *code of size N* a sequence $(u_1, X_1), \dots, (u_N, X_N)$, where u_k are words of length n , and $\{X_k\}$ is a family of positive operators in $\mathcal{H}^{\otimes n}$, satisfying $\sum_{j=1}^N X_j \leq I$. Defining $X_0 = I - \sum_j X_j$, we have a resolution of identity in $\mathcal{H}^{\otimes n}$. An output $k \geq 1$ means decision that the word u_k was transmitted, while the output 0 is interpreted as evasion of any decision. The average error probability for such a code is

$$\bar{\lambda} = \frac{1}{N} \sum_{k=1}^N [1 - \text{Tr} S_{u_k} X_k]. \quad (6)$$

Let us denote $p(n, N)$ the minimum of this error probability with respect to all codes of the size N with words of length n , satisfying the condition

$$f(i_1) + \dots + f(i_n) \leq nE. \quad (7)$$

As a direct consequence of theorems in Sec. 7.3 of [4], one can prove the following statement providing information-theoretic justification of the definition (5):

Proposition 1. *If $R < C$, then $p(n, e^{nR}) \rightarrow 0$, and if $R > C$, then $p(n, e^{nR}) \not\rightarrow 0$.*

Let $H(S) = -\text{Tr} S \log S$ be the von Neumann entropy of a d. o. S and let $\pi = \{\pi_i\}$ be an apriori distribution on A . We denote

$$\bar{S}_\pi = \sum_{i \in A} \pi_i S_i$$

and assume that

$$\sup_{\pi \in \mathcal{P}_1} H(\bar{S}_\pi) < \infty. \quad (8)$$

Further, we denote

$$\Delta H(\pi) = H(\bar{S}_\pi) - \sum_{i \in A} \pi_i H(S_i). \quad (9)$$

Proposition 2. *Under the condition (8)*

$$C = \sup_{\pi \in \mathcal{P}_1} \Delta H(\pi). \quad (10)$$

Proof. To prove the \leq part, take $\pi \in \mathcal{P}_n$. The *entropy bound* [9], [16] implies

$$I_n(\pi, X) \leq \Delta H_n(\pi),$$

where $\Delta H_n(\pi)$ is the analog of $\Delta H(\pi)$ for the product channel. According to the subadditivity property of quantum entropy [1],

$$\Delta H_n(\pi) \leq \sum_{k=1}^n \Delta H(\pi^{(k)}),$$

where $\pi^{(k)}$ is the k -th marginal distribution of π on A . Therefore

$$\frac{1}{n} I_n(\pi, X) \leq \frac{1}{n} \sum_{k=1}^n \Delta H(\pi^{(k)}) \leq \Delta H(\bar{\pi}),$$

where $\bar{\pi} = \frac{1}{n} \sum_{k=1}^n \pi^{(k)}$, since $\Delta H(\pi)$ is concave function of π [9]. Also inequality (3) can be rewritten as

$$\sum_{k=1}^n \sum_{i_k} f(i_k) \pi^{(k)}(i_k) \leq nE.$$

It follows that $n^{-1}C_n \leq \sup_{\pi \in \mathcal{P}_1} \Delta H(\pi)$ and hence, a similar inequality holds for C .

To prove the \geq part, we use the random coding modified for constrained inputs. Let π be a distribution satisfying (2), and let \mathbf{P} be a distribution on the set of M words, under which the words are independent and

$$\mathbf{P}(u = (i_1, \dots, i_n)) = \pi_{i_1} \cdot \dots \cdot \pi_{i_n}.$$

Let $\nu_n = \mathbf{P}(\frac{1}{n} \sum_{k=1}^n f(i_k) \leq E)$ and define the modified distribution

$$\tilde{\mathbf{P}}(u = (i_1, \dots, i_n)) = \begin{cases} \nu_n^{-1} \pi_{i_1} \cdot \dots \cdot \pi_{i_n}, & \text{if } \sum_{k=1}^n f(i_k) \leq nE, \\ 0, & \text{otherwise.} \end{cases}$$

Let us remark that if $\pi \in \mathcal{P}_1$, then $\mathbf{M}f \leq E$ (where \mathbf{M} ($\tilde{\mathbf{M}}$) is the expectation corresponding to \mathbf{P} ($\tilde{\mathbf{P}}$)) and hence by the central limit theorem

$$\lim_{n \rightarrow \infty} \nu_n \geq 1/2.$$

Therefore $\tilde{M}\xi \leq 2^m M\xi$ for any nonnegative random variable ξ depending on m words. In particular, for the error probability (6) we gave in [13] the upper bound (17) depending on arbitrary two words, the expectation of which with respect to \mathbf{P} can be made arbitrarily small provided $M = e^{nR}, n \rightarrow \infty$, with $R < C$. Thus $\tilde{M}\bar{\lambda}$ also can be made arbitrarily small under the same circumstances. The proof in [13] is for finite dimensional Hilbert space, but under the condition (8) it can be modified for infinite dimensions. Since the distribution $\tilde{\mathbf{P}}$ is concentrated on words satisfying (7), we can choose a code satisfying this constraint for which $\bar{\lambda}$ can be made arbitrarily small. Proposition 1 then implies that

$$C \geq \sup_{\pi \in \mathcal{P}_1} \Delta H(\pi),$$

which completes the proof.

II. THE CASE OF CONTINUOUS ALPHABET

In this section we take as the input alphabet A arbitrary Borel subset in a finite-dimensional Euclidean space \mathcal{E} . We assume that the channel is given by *weakly continuous* mapping $x \rightarrow S_x$ from the input alphabet A to the set of density operators in \mathcal{H} . (The weak continuity means continuity of all matrix elements $\langle \psi | S_x \phi \rangle; \psi, \phi \in \mathcal{H}$). We assume that a *continuous* function f on \mathcal{E} is fixed and consider the set \mathcal{P}_1 of probability measures π on A satisfying

$$\int_A f(x) \pi(dx) \leq E. \quad (11)$$

Like in the classical case, we discretize the channel by taking apriori distributions with discrete supports

$$\pi(dx) = \sum_i \pi_i \delta_{x_i}(dx), \quad (12)$$

where $\{x_i\} \subset A$ is arbitrary countable collection of points and

$$\delta_x(B) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B, \end{cases}$$

and by taking discrete resolutions of identity $\{X_j\}$. For π of the form (12) the constraint (11) takes the form (2). Then we define the capacity of the

channel $x \rightarrow S_x$ with the constraint (11) by repeating the argument in Sec. 1 with the only modifications that

$$\mathbf{P}(j|i) = \text{Tr} S_{x_i} X_j,$$

and additional supremum in (4) is taken over all possible choices of the points $x_i \in A$.

For arbitrary $\pi \in \mathcal{P}_1$ consider the quantity

$$\Delta H(\pi) = H(\bar{S}_\pi) - \int_A H(S_x) \pi(dx), \quad (13)$$

where

$$\bar{S}_\pi = \int_A S_x \pi(dx). \quad (14)$$

Because of the weak continuity of the function S_x the integral is well defined and represents a density operator in \mathcal{H} . Moreover, from the Lemma below it follows that the nonnegative function $H(S_x)$ is lower semicontinuous, and hence the second term in (13) is also well defined.

Proposition 3. *Under the condition (8), in which \bar{S}_π is given by (14) and \mathcal{P}_1 – by (11),*

$$C = \sup_{\pi \in \mathcal{P}_1} \Delta H(\pi). \quad (15)$$

Proof. The \leq part of the proof follows obviously from Proposition 2, as for π given by (12) the quantity (13) turns into (9). To prove the \geq part it is sufficient to construct, for arbitrary $\pi \in \mathcal{P}_1$, a sequence of discrete $\pi^{(l)} \in \mathcal{P}_1$ such that

$$\liminf_{l \rightarrow \infty} \Delta H(\pi^{(l)}) \geq \Delta H(\pi). \quad (16)$$

To this end for any $l = 1, \dots$ we consider the division of A into disjoint subsets

$$B_k^{(l)} = \{x : k/l \leq H(S_x) < (k+1)/l\}, \quad k = \dots, -1, 0, 1, \dots \quad (17)$$

By making, if necessary, a finer subdivision, we can always assume that diameters of all sets $B_k^{(l)}$ are bounded from above by ϵ_l , where $\epsilon_l \rightarrow 0$ as $l \rightarrow \infty$. Let $x_k^{(l)}$ be a point at which $f(x)$ achieves its minimum on the closure $\bar{B}_k^{(l)}$ of $B_k^{(l)}$, and define

$$\pi^{(l)}(dx) = \sum_k \pi(B_k^{(l)}) \delta_{x_k^{(l)}}(dx), \quad (18)$$

where π is a fixed distribution from \mathcal{P}_1 . Then

$$\int_A f(x) \pi^{(l)}(dx) \leq \int_A f(x) \pi(dx),$$

hence $\pi^{(l)} \in \mathcal{P}_1$.

By construction (17), (18) and due to the condition (8) we have

$$\left| \int_A H(S_x) \pi^{(l)}(dx) - \int_A H(S_x) \pi(dx) \right| \leq 1/l,$$

and it remains to show that

$$\liminf_{l \rightarrow \infty} H \left(\int_A S_x \pi^{(l)}(dx) \right) \geq H \left(\int_A S_x \pi(dx) \right). \quad (19)$$

We first remark that due to the weak continuity and uniform boundedness of the function S_x , the density operators $\int_A S_x \pi^{(l)}(dx)$ weakly converge to the density operator $\int_A S_x \pi(dx)$. Indeed, let B_c be the ball of radius c in \mathcal{E} . Then

$$\begin{aligned} & \left| \langle \phi | \int_A S_x \pi^{(l)}(dx) \psi \rangle - \langle \phi | \int_A S_x \pi(dx) \psi \rangle \right| \\ & \leq \sum_k \int_{B_k^{(l)} \cap B_c} | \langle \phi | S_{x_k^{(l)}} \psi \rangle - \langle \phi | S_x \psi \rangle | \pi(dx) + 2 \|\phi\| \|\psi\| \pi(A \setminus B_c). \end{aligned}$$

By choosing first c large enough to make the second term small, we can make the first term small for all large enough l since $\langle \phi | S_x \psi \rangle$ is uniformly continuous on $A \cap B_c$ and the diameters of $B_k^{(l)}$ uniformly tend to zero.

It remains to apply the following Lemma (this result is well known but we include sketch of its proof for completeness):

Lemma. *Let $\{A_l\}$ be a sequence of density operators, weakly converging to a density operator A . Then*

$$\liminf_{l \rightarrow \infty} H(A_l) \geq H(A).$$

Proof. Let $\{P_m\}$ be a monotonely increasing sequence of finite-dimensional projections weakly converging to unit operator in \mathcal{H} . By Lemma 4 from [15] the sequence

$$H(P_m S P_m) + \text{Tr} P_m S P_m \log \text{Tr} P_m S P_m$$

monotonely converges to $H(S)$ for any d. o. S . Then we have

$$\liminf_{l \rightarrow \infty} H(A_l) = \liminf_{l \rightarrow \infty} \lim_{m \rightarrow \infty} H(P_m A_l P_m) \geq \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} H(P_m A_l P_m) = H(A).$$

This completes the proof of (16) and hence of Proposition 3.

III. THE QUANTUM GAUSSIAN CHANNEL WITH CONSTRAINED ENERGY OF THE SIGNAL

Let A be the complex plane \mathbf{C} , and let for every $\alpha \in \mathbf{C}$ the density operator S_α describe the thermal state of harmonic oscillator with the signal amplitude α and the mean number of the noise quanta N , i. e.

$$S_\alpha = \frac{1}{\pi N} \int \exp\left(-\frac{|z - \alpha|^2}{N}\right) |z\rangle\langle z| d^2 z, \quad (20)$$

where $|z\rangle$ are the coherent state vectors. This is quantum analog of channel with additive Gaussian noise (see [6], [8], [10], [3]). We remind for future use that

$$S_\alpha = V(\alpha) S_0 V(\alpha)^*,$$

where

$$V(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a)$$

are the unitary displacement operators, a^\dagger, a being the creation - annihilation operators for the harmonic oscillator, and the operator S_0 has the spectral representation:

$$S_0 = \frac{1}{N+1} \sum_{n=0}^{\infty} \left(\frac{N}{N+1}\right)^n |n\rangle\langle n|, \quad (21)$$

where $|n\rangle$ are the eigenvectors of the number operator $a^\dagger a$. The states (20) all have the same entropy

$$H(S_\alpha) = (N+1) \log(N+1) - N \log N, \quad (22)$$

and the mean number of quanta

$$\text{tr} S_\alpha a^\dagger a = N + |\alpha|^2. \quad (23)$$

We impose the following constraint onto the mean energy of the signal

$$\int |\alpha|^2 \pi(d^2 \alpha) \leq E, \quad (24)$$

where $\pi(d^2 \alpha)$ is an apriori distribution. In fact, E is the “mean number of quanta” in the signal, which is proportional to energy for one mode. Consider the density operator

$$\bar{S}_\pi = \int S_\alpha \pi(d^2 \alpha).$$

The constraint (24) by virtue of (23) implies

$$\text{Tr } \bar{S}_\pi a^\dagger a \leq N + E. \quad (25)$$

It is well known that under this constraint the maximal entropy

$$H(\bar{S}_\pi) = (N + E + 1) \log(N + E + 1) - (N + E) \log(N + E) \quad (26)$$

is attained by Gaussian density operator

$$\bar{S}_\pi = \frac{1}{\pi(N + E)} \int \exp\left(-\frac{|z|^2}{(N + E)}\right) |z\rangle\langle z| d^2 z, \quad (27)$$

corresponding to the optimal apriori distribution

$$\pi(d^2 \alpha) = \frac{1}{\pi E} \exp\left(-\frac{|\alpha|^2}{E}\right) d^2 \alpha. \quad (28)$$

Hence the condition of Proposition 3 is fulfilled, and the capacity of the channel is equal to

$$\begin{aligned} C = H(\bar{S}_\pi) - H(S_\alpha) &= \log\left(1 + \frac{E}{N + 1}\right) \\ &+ (N + E) \log\left(1 + \frac{1}{N + E}\right) - N \log\left(1 + \frac{1}{N}\right). \end{aligned}$$

This quantity was anticipated in [6] (relation (4.28)) as an upper bound for the information transmitted by the quantum Gaussian channel. On the other hand, for a long time this quantity was also known as the capacity of the “narrow band photon channel” [5], [14]. Our argument based on Proposition 3 gives for the first time the proof of the asymptotic equivalence, in the sense of information capacity, of the Gaussian channel with the energy constraint (24) to this quasiclassical channel. To make the point clear, we give below a simplified one-mode description of the photon channel.

Consider the discrete family of states

$$S_m = P(m) S_0 P(m)^*, \quad m = 0, 1, \dots, \quad (29)$$

where $P(m)$ is energy shift operator satisfying $P(m)|n\rangle = |n + m\rangle$. Notice that $P(m) = P^m$, where P is isometric operator adjoint to the quantum-mechanical “phase operator” [12]. The states S_m all have the same entropy (22) as the states S_α , and the mean number of quanta

$$\text{tr} S_m a^\dagger a = N + m. \quad (30)$$

Moreover, all states (29) are diagonal in the number representation, in which sense the channel may be called quasiclassical.

Imposing the constraint

$$\sum_{m=0}^{\infty} m \pi_m \leq E, \quad (31)$$

where π_m is an apriori distribution, and introducing the density operator

$$\bar{S}'_{\pi} = \sum_{m=0}^{\infty} \pi_m S_m,$$

by virtue of (30), we obtain the same constraint (25) for the new operator \bar{S}'_{π} . The maximal entropy (26) is again attained by the operator (27), which has the spectral representation

$$\bar{S}_{\pi} = \frac{1}{N + E + 1} \sum_{n=0}^{\infty} \left(\frac{N + E}{N + E + 1} \right)^n |n\rangle\langle n|. \quad (32)$$

It corresponds to the optimal apriori distribution [14]

$$\pi_m = \frac{N}{N + E} \delta_{m0} + \frac{E}{N + E} \left[\frac{1}{N + E + 1} \left(\frac{N + E}{N + E + 1} \right)^m \right].$$

There is notable difference between the case of pure-state channel as opposed to the general case. For a pure-state case (where $N = 0$), one can formulate a broader problem of finding a maximum capacity channel $x \rightarrow S_x$ with arbitrary alphabet $\{x\}$ and an apriori distribution $\pi(dx)$ satisfying the output constraint

$$\text{Tr} \bar{S}_{\pi} a^{\dagger} a \leq E.$$

This was done in [16] where it was shown that the noiseless photon channel provides a solution to this problem. In view of the result of [7], any other pure-state channel satisfying

$$\int S_x \pi(dx) = \frac{1}{E + 1} \sum_{n=0}^{\infty} \left(\frac{E}{E + 1} \right)^n |n\rangle\langle n|$$

gives, asymptotically, a solution to the same problem. However, in the general case imposing the output constraint (25) instead of the input constraints (24)

or (31) looks rather artificial; the equivalence of these constraints for apparently different channels seems to be a very special feature of the quantum Gaussian density operators.

IV. THE UPPER BOUNDS FOR ERROR PROBABILITY

A much more detailed information concerning the rate of convergence of the error probability can be obtained for pure-state channels, by modifying the estimates from [2] to channels with infinite alphabets and constrained inputs following the method of [4], Ch. 7. We start with the case of discrete alphabet.

Let $S_i = |\psi_i\rangle\langle\psi_i|$ be the pure signal states of the channel, and let π be an apriori distribution satisfying the restriction (2). Then the following *random coding bound* holds for the error probability $p(n, N)$ where $N = e^{nR}$ with $R < C$:

$$p(n, e^{nR}) \leq 2 \left(\frac{e^{p\delta}}{\nu_{n,\delta}} \right)^2 \exp\{-n[\mu(\pi, s, p) - sR]\}, \quad (33)$$

where

$$\mu(\pi, s, p) = -\log \text{Tr} \left\{ \sum_i \pi_i e^{p[f(i)-E]} S_i \right\}^{1+s}, \quad (34)$$

and $0 \leq s \leq 1, 0 \leq p, 0 < \delta$ are arbitrary parameters. The quantity

$$\nu_{n,\delta} = \text{P}(En - \delta \leq \sum_{k=1}^n f(i_k) \leq nE)$$

satisfies $\lim_{n \rightarrow \infty} \sqrt{n} \nu_{n,\delta} > 0$, thus adding only $o(n)$ to the exponential in (33).

The bound (33) is obtained in the same way as Proposition 1 in [2], that is by evaluating the expectation of the average error probability (6) using random, independently chosen codewords, but with the modified codeword distribution

$$\tilde{\text{P}}_\delta(u = (i_1, \dots, i_n)) = \begin{cases} \nu_{n,\delta}^{-1} \pi_{i_1} \cdot \dots \cdot \pi_{i_n}, & \text{if } nE - \delta \leq \sum_{k=1}^n f(i_k) \leq nE, \\ 0, & \text{otherwise.} \end{cases} \quad (35)$$

The point is that for any random variable ξ depending on m words

$$\tilde{M}_\delta \xi \leq \left(\frac{e^{p\delta}}{\nu_{n,\delta}} \right)^m \text{Mexp} \left\{ mp \sum_{k=1}^n [f(i_k) - E] \right\} \xi,$$

where $p \geq 0$. By using this inequality after equation (14) in the proof of Proposition 1 from [2], and following argument in Ch. 7 of [4], we can obtain the bound (33).

In the same way, the proof of Proposition 2 from [2] can be modified to obtain the *expurgated bound*

$$p(n, e^{nR}) \leq \exp \left\{ -n [\tilde{\mu}(\pi, s, p) - s(R + \frac{2}{n} \log \frac{2e^{p\delta}}{\nu_{n,\delta}})] \right\}, \quad (36)$$

where

$$\tilde{\mu}(\pi, s, p) = -s \log \sum_{i,k} \pi_i \pi_k e^{p[f(i)+f(k)-2E]} | \langle \psi_i | \psi_k \rangle |^{2/s}. \quad (37)$$

These bounds can be extended to pure-state channels with continuous alphabets by using technique of Sec. II to obtain (33), (36) with

$$\mu(\pi, s, p) = -\log \text{Tr} \left\{ \int_A e^{p[f(x)-E]} S_x \pi(dx) \right\}^{1+s}, \quad (38)$$

$$\tilde{\mu}(\pi, s, p) = -s \log \int_A \int_A e^{p[f(x)+f(y)-2E]} | \langle \psi_x | \psi_y \rangle |^{2/s} \pi(dx) \pi(dy). \quad (39)$$

Introducing the *reliability function*

$$E(R) = \lim_{n \rightarrow \infty} \sup \frac{1}{n} \log \frac{1}{p(n, e^{nR})},$$

which characterizes the exponential rate of convergence of the error probability, we get the lower bound for $E(R)$:

$$E(R) \geq \max \{ E_r(R), E_{ex}(R) \},$$

where

$$E_r(R) = \max_{0 \leq s \leq 1} (\max_{0 \leq p} \max_{\pi \in \mathcal{P}_1} \mu(\pi, s, p) - sR), \quad (40)$$

$$E_{ex}(R) = \max_{1 \leq s} (\max_{0 \leq p} \max_{\pi \in \mathcal{P}_1} \tilde{\mu}(\pi, s, p) - sR). \quad (41)$$

An example where the maximization at least partially can be performed analytically is considered in the following Section.

V. THE RELIABILITY FUNCTION OF QUANTUM GAUSSIAN PURE-STATE CHANNEL

We are going to apply results of the previous Section to the Gaussian pure-state channel $\alpha \rightarrow S_\alpha = |\alpha\rangle\langle\alpha|$ with the constraint (24). By taking the optimal apriori distribution (28) we can calculate explicitly the functions (38), (39).

Namely, to calculate (38), we remark that

$$\begin{aligned} \int e^{p(|z|^2 - E)} S_z \pi(d^2 z) &= \frac{e^{-pE}}{1 - pE} \frac{1}{\pi E'} \int e^{-\frac{|z|^2}{E'}} |z\rangle\langle z| d^2 z \\ &= \frac{e^{-pE}}{1 - pE} \frac{1}{E' + 1} \sum_{n=0}^{\infty} \left(\frac{E'}{E' + 1} \right)^n |n\rangle\langle n|, \end{aligned}$$

where $E' = E/(1 - pE)$, provided $p < E^{-1}$, and the trace of the $(1 + s)$ -th power of this operator is easily calculated to yield

$$\mu(\pi, s, p) = (1 + s)pE + \log[(1 + E - pE)^{1+s} - E^{1+s}]. \quad (42)$$

By taking into account that

$$|\langle z|w\rangle|^2 = e^{-|z-w|^2},$$

(see, e. g. [8]), we can calculate the integral in (39) as

$$\begin{aligned} \frac{e^{-2pE}}{(\pi E)^2} \int \int \exp\{ -[(E^{-1} + s^{-1} - p)|z|^2 + (E^{-1} + s^{-1} - p)|w|^2 - 2s^{-1}\text{Re}\bar{z}w] \} \\ = \frac{e^{-2pE}}{1 + p^2 E^2 - 2pE - 2pE^2/s + 2E/s}, \end{aligned}$$

for $p < E^{-1}$, whence

$$\tilde{\mu}(\pi, s, p) = s\{2pE + \log[1 + p^2 E^2 - 2pE + 2E(1 - pE)/s]\}. \quad (43)$$

Trying to maximize $\mu(\pi, s, p)$ with respect to p we obtain the equation

$$(1 + E - pE)^s (1 - p) = E^s, \quad (44)$$

which can be solved explicitly only for $s = 0, 1$. Thus, contrary to the classical case [4], the maximum in (40) in general can be found only numerically. For $s = 0$ we have $p = 0$ and

$$C = \frac{\partial}{\partial s} \mu(\pi, 0, 0) = (E + 1) \log(E + 1) - E \log E.$$

For $s = 1$ equation (44) has the unique solution $p(1, E) = 1 + 1/E - g(E)/E < E^{-1}$, where

$$g(E) = \frac{1 + \sqrt{4E^2 + 1}}{2}.$$

For future use we find the important quantities

$$\begin{aligned} \mu(\pi, 1, p(1, E)) &= 2(E + 1 - g(E)) + \log g(E); \\ \frac{\partial}{\partial s} \mu(\pi, 1, p(1, E)) &= E + 1 - g(E) + \frac{g(E)^2 \log g(E) - E^2 \log E}{g(E)^2 - E^2}. \end{aligned} \quad (45)$$

The optimization of the expurgated bound can be performed analytically. Taking partial derivative with respect to p we obtain the equation

$$p^2 - 2p \left(\frac{1}{s} + \frac{1}{2E} \right) + \frac{1}{sE} = 0,$$

the solution of which, satisfying $p < E^{-1}$, is

$$p(s, E) = s^{-1} + E^{-1} - E^{-1}g(E/s).$$

Substituting this in (41), we obtain the following expression, which is to be maximized with respect to $s \geq 1$:

$$\tilde{\mu}(\pi, s, p(s, E)) - sR = 2(E + s - sg(E/s)) + s \log g(E/s) - sR.$$

Taking derivative with respect to s , we obtain the equation

$$g(E/s) = e^R,$$

the solution of which is

$$s = \frac{E}{\sqrt{e^{2R} - e^R}}. \quad (46)$$

If this is less than 1, which is equivalent to

$$R < \log g(E) = \frac{\partial}{\partial s} \tilde{\mu}(\pi, 1, p(1, E)),$$

then the maximum is achieved for the value of s given by (46) and is equal to

$$2E(1 - \sqrt{1 - e^{-R}}) = E_{ex}(R) > E_r(R),$$

(which up to a factor coincides with the expurgated bound for classical Gaussian channel). In the range

$$\frac{\partial}{\partial s} \tilde{\mu}(\pi, 1, p(1, E)) \leq R \leq \frac{\partial}{\partial s} \mu(\pi, 1, p(1, E)),$$

where the optimizing s is equal to 1, we have the linear bound

$$E_{ex}(R) = E_r(R) = \mu(\pi, 1, p(1, E)) - R,$$

with the quantities $\frac{\partial}{\partial s} \mu(\pi, 1, p(1, E)), \mu(\pi, 1, p(1, E))$ defined by (45). Finally, in the range

$$\frac{\partial}{\partial s} \mu(\pi, 1, p(1, E)) < R < C$$

we have $E_{ex}(R) < E_r(R)$ with $E_r(R)$ given implicitly by (40).

On the other hand, for the pure-state photon channel the analysis of the error probability is trivial: since this is quasiclassical noiseless channel, the error probability is zero for $R < C$. Thus, although the two channels are asymptotically equivalent in the sense of capacity, their finer asymptotic properties are apparently essentially different.

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